ELECTORAL OUTCOMES AND SOCIAL LOG-LIKELIHOOD MAXIMA

by

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ABSTRACT

This paper is concerned with the public policies that occur in economies with elections when political candidates estimate voting behavior with log-concave probabilistic voting estimators (e.g., normal estimators). We establish that, for a vector of policies to be the outcome of an election, it is both necessary and sufficient that these policies maximize the society's mean (or social) log-likelihood function. This implies: First, the set of possible electoral outcomes is convex. Second, there is an electoral equilibrium whenever the set of social alternatives is compact. This property which holds for all multi-dimensional policy spaces does not use any special symmetry requirements on voter preferences. Third, under "cardinal probabilistic voting," every electoral outcome is also a maximum of a Nash type Social Welfare function. Fourth, in a finite population of m voters with independent probabilistic voting density functions a vector of policies is an electoral outcome if and only if it has the maximum estimated likelihood of receiving unanimous support.

ELECTORAL OUTCOMES AND SOCIAL LOG-LIKELIHOOD MAXIMA*

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Peter Coughlin** and Shmuel Nitzan***

1. Introduction

This paper extends the foundations of the spatial model of electoral competition with probabilistic voting developed by Coughlin and Nitzan [1979] from the earlier work of Comaner [1976], Hinich ([1977], [1978]), Hinich, et.al. [1972] and Kramer [1976]. In particular, this paper studies the public policies that occur in economies with elections when political candidates estimate voting behavior with log-concave probabilistic voting estimators (e.g., normal estimators).

The electoral competitions analyzed in this paper are described in Section 2. We then establish (in Section 3) that, for a vector of policies to be the outcome of an election, it is both necessary and sufficient that these policies maximize the society's mean log-likelihood function. This analysis, therefore, shows that these two alternative social choice mechanisms are, in fact, equivalent.

This result provides us with a useful tool for analyzing elections in the kind of societies studied in this paper. In particular, we have derived the following implications from the main theorem. First,

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the set of possible electoral outcomes is convex. Second, an electoral equilibrium exists when the set of social alternatives is compact. (The theorem also provides a very useful method for finding these equilibria.) Third, when there is "cardinal probabilistic voting," every electoral outcome is a maximum of a Nash type Social Welfare function. (This extends the main result in Coughlin and Nitzan [1979].) Fourth, in a finite population of m voters with equally likely independent types of probabilistic voting behavior a vector of policies is an electoral outcome if and only if it has the maximum estimated likelihood of receiving unanimous support.

2. Probabilistic Voting and Electoral Competition

Empirical studies of voting behavior as a function of proposed policies and existing economic conditions leave a significant amount of unexplained variation (e.g., Kramer [1971], Stigler [1973], Arcelus and Meltzer [1975], Bloom and Price [1975] and Fair [1978]). This has led to the conclusion that uncertainty and non-policy considerations result in random (or indeterminate) voting behavior when this behavior is viewed as a function of existing and proposed policies.

Intriligator ([1973], [1979]), Fishburn [1975] and Fishburn and Gehrlein [1977] have formulated the indeterminateness in voter behavior with individual choice probabilities. For a Euclidean policy space, these individual choice probabilities are summarized by a density function on $X \subset R^n$ (e.g., Nitzan [1975]). We assume that X is nonempty and convex. The <u>probabilistic voting density function</u> of each individual expresses the probabilities of his choosing an alternative in differenct possible subsets of X--given that he can determine the social choice unilaterally.

Learning the behavior of every individual is impossible. Therefore, political entrepreneurs (or candidates) have to estimate voter behavior. Since the candidates usually have access to the same information (polls, past election data, etc.), we will assume that candidates obtain a common probabilistic voting estimator which estimates the proportions of the population that are described by particular probabilistic voting density functions. To be precise, Let $\Theta \subseteq R^{\ell}$ denote an index set of parameters for a class of density functions. Then we are assuming that the candidates obtain a probabilistic voting estimator, $\hat{g}(\theta)$, which is a density function on Θ.

For instance, candidates may be willing to use normal or truncated normal probabilistic voting estimators. In this case, the candidates can estimate the proportions of the population that fall into certain combinations of possible means and variances.

When the individual probabilistic voting density functions are known, there is no estimation problem. This case was studied in Nitzan [1975] and Coughlin and Nitzan [1979]. $f(x;\theta)$ will denote a real-valued density function on X which has the parameter $\theta \in \Theta$. We will assume that $f(x;\theta)$ is log-concave in x (i.e., that $\log f(x;\theta)$ is a concave function of x; see, for example, Roberts and Varberg [1973]).

This is strictly weaker than assuming concavity for the $f(x;\theta)$ (as in Coughlin and Nitzan [1979]), but also strictly stronger than assuming quasi-concavity.

Since these density functions are estimates of individual behavior, we will make three regularity assumptions:

(i) Each $f(x;\theta)$ is positive and continuously differentiable everywhere on X.

(ii) At each $x \in X$, $f(x;\theta)$ is a bounded, continuous function of θ.

(iii) At each $x \in X$, $\partial f(x;\theta)/\partial x_h$ $(h = 1,\ldots,n)$, is a bounded continuous function of θ.

This class of functions contains many commonly used density functions including the normal and truncated normal.

An electoral competition is a game with two players (the candidates), a policy space (the strategy space for both candidates) and a payoff function for each player defined on pairs of proposed policies. We will let $\Psi_j \in X$ denote the policy proposed by candidate j, j = 1,2. Additionally, $P_\theta^1(\Psi_1,\Psi_2)$ will denote the probability that an individual whose behavior is described by $f(x;\theta)$ votes for candidate j when Ψ_1 and Ψ_2 are proposed by candidates 1 and 2, respetively. The following three assumptions provide the basis for our calculation of the payoff functions.

First, we assume that the candidates estimate the behavior of concerned citizens who vote. This means that we are concerned with a <u>full participation electorate</u> where

$$(1) \qquad P_\theta^1(\Psi_1,\Psi_2) + P_\theta^2(\Psi_1,\Psi_2) = 1$$

for every $\theta \in \Theta$ and $(\Psi_1,\Psi_2) \in X \times X$. This assumption is repeatedly made in the literature on voting equilibria (e.g., see Comaner [1976], Hinich ([1977], [1978]), Kramer [1978] and Coughlin and Nitzan [1979]).

Second, we want to relate random voting behavior on a Euclidean policy space to behavior on binary choices, i.e., choices between two proposed policies. We therefore assume that each individual's choice probabilities satisfy <u>independence from irrelevant alternatives</u>:

$$(2) \qquad \frac{P_\theta^1(\Psi_1,\Psi_2)}{P_\theta^2(\Psi_1,\Psi_2)} = \frac{f(\Psi_1;\,\theta)}{f(\Psi_2;\,\theta)}$$

for each $\theta \in \Theta$ and $(\Psi_1,\Psi_2) \in X \times X$.

This merely states that the relative likelihoods of choosing Ψ_1 and Ψ_2 from X are preserved when choosing from $\{\Psi_1,\Psi_2\}$. This is the continuous version of the independence from irrelevant alternatives which follows from the basic choice axioms in Luce [1959] (Axiom 1 and Lemma 3) (also in Luce and Suppes [1965] and Ray [1973]).

Finally, since vote totals are random, a candidate could choose to maximize his expected plurality or his probability of winning. Hinich ([1977], pp. 212-213) argues that for a large electorate with a reasonable large amount of indeterminateness these two objectives are equivalent. We therefore prefer the more tractable objective function and assume that each candidate has the objective of maximizing his expected plurality.

$P(\Psi_1,\Psi_2)$ will denote the expected plurality for candidate 1 when Ψ_1 and Ψ_2 are proposed by candidates 1 and 2 respectively. $-P(\Psi_1,\Psi_2)$ is then the expected plurality for candidate 2. The expected plurality electoral competition is therefore given by the game

$$(3) \qquad \Gamma = \Gamma(X,X,(P(x_1,x_2), - P(x_1,x_2)))$$

We will denote the class of games which satisfy the assumptions of this section by Γ_2. Γ_1 was defined in Coughlin and Nitzan [1979] as the class of expected plurality electoral competitions in societies which have a finite set of voters and cardinal concave probabilitic voting behavior (see p. 8) which is known to the candidates.

3. Electoral Equilibria and Social Choice Probabilities

An <u>electoral equilibrium</u> (in pure strategies) is any pair, $(x^*,y^*) \in X \times X$, such that

$$(4) \qquad P(x,y^*) \leq P(x^*,y^*) \leq P(x^*,y)$$

for every $x,y \in X$. Any policy, $x \in X$, which is in some electoral equilibrium pair will be called an <u>electoral outcome</u> (since either policy in the pair may be the winning position and, hence, adopted by society).

There are many alternatives to the existing institution of political elections which have proposed for aggregating individual choice probabilities (e.g., see Intrilligator [1973], Fishburn [1975] and Nitzan [1975]). We will show that holding an election is equivalent to one such possible aggregation scheme.

Recall that individual choice probabilities are summarized in $f(x;\ \theta)$, or, equivalently in the log-likelihood function $\ln f(x;\ \theta)$. The society's <u>mean (or social) log-likelihood function</u> is therefore given by

$$L(x) = \int_{\Theta} \ln f(x;\ \theta) \cdot \hat{g}(\theta) \cdot d\theta \tag{5}$$

for $x \in X$.

Using this terminology, we are now able to state our main result:

<u>Theorem</u>: $x \in X$ is an electoral outcome of an expected plurality competition in Γ_2 if, and only if, x is a maximum of the society's mean log-likelihood function.

This theorem completely characterizes the set of policies which can occur as electoral outcomes. Put differently, it shows that the two alternative social choice mechanisms defined above are, in fact equivalent. In addition, this result reduces the complex two-candidate decision problem to an optimization problem with a single vector of decision variables. This maximization problem can be solved more easily than the original two candidate game with infinite strategy sets.

By using this equivalent maximization problem to analyze the election games in Γ_2, we have two immediate corollaries:

Corollary 1: The set of electoral outcomes in any expected plurality electoral competition in Γ_2 is convex.

Corollary 2: Suppose X is compact. Then there is an electoral equilibrium for every expected plurality electoral competition in Γ_2.

This second corollary is especially interesting since (i) most studies of social choice problems are concerned with compact sets of social alternatives, and (ii) a considerable portion of this literature is devoted to the development of sufficient conditions for the existence of voting equilibria.

Corollary 2 is also interesting for another reason. Since X is convex (Section 2) and $P(x,\Psi)$ is log-concave (Lemma 2) and, hence, quasi-concave in x, the existence of electoral equilibria (when X is compact) follows by a standard fixed point theorem. However, we have established this result by an alternative argument which has provided us with a constructive method for finding the game's equilibria (which is an alternative to fixed point algorithms). This method is similar to the gradient methods developed in Arrow, Hurwicz and Uzawa [1958].

Elsewhere (Coughlin and Nitzan [1979]) we have studied the special case of "cardinal probabilistic voting" where there is a ratio-scale utility function $U_\theta(x)$ for each individual which satisfies

$$(6) \qquad f(x;\theta) = U_\theta(x)$$

(see Coughlin and Nitzan [1979], Luce [1959], and Fishburn [1978]). Under this condition our theorem reveals a connection between electoral competitions and a generalization of a Nash type social welfare function for infinite populations. In the finite population case where the individuals are indexed by $i = 1,\dots,m$, the Nash social welfare function is given by

$$(7) \qquad W(x) = \sum_{i=1}^{m} \log (U_i(x))$$

(see, for instance, Kaneko and Nakamura [1979]).

When a society has a large population, whose preferences are estimated by a continuous density function $(\hat{g}(\theta))$ on an index set $(\Theta \subset R^{\ell})$, (7) generalizes to

$$(8) \qquad W(x) = \int_{\Theta} \log U_\theta(x) \cdot \hat{g}(\theta) \cdot d\theta$$

(8) is the society's <u>estimated Nash Social Welfare function</u>.

The main theorem immediately implies:

<u>Corollary 3</u>: Suppose that $\Gamma \in \Gamma_2$ has cardinal probabilistic voting. Then $x \in X$ is an electoral outcome in Γ if, and only if, x is a maximum of the society's estimated Nash Social Welfare function.

This extends the earlier result of Coughlin and Nitzan ([1979], Theorem 1) which showed that, when there is a finite number of citizens with cardinal probabilicitic voting which is known to candidates, an alternative is an electoral outcome if and only if it is a global maximum of (7).

Finally, suppose there are m voters with equally likely independent probabilistic voting density functions $f(x;\theta_i)$ $i = ,\ldots,m$. In this special case the estimated likelihood that a policy x receives unanimous support when voters can choose any policy in x is

(9) $$L'(x) = \prod_{i=1}^{m} f(x;\theta_i) \quad .$$

A maximum of $L'(x)$ will be referred to as a <u>unanimity likelihood maximum</u>. We can now state

Corollary 4: Consider a finite population of m voters with equally likely independent probabilistic density functions. Then $x \in X$ is an electoral outcome if, and only if, it is a unanimity likelihood maximum.

4. Conclusion

This paper has extended the foundations of spatial models of electoral competition to include large populations, log-concave probabilistic voting and estimators for candidates. The main theorem gives a necessary and sufficient condition for a policy to be an electoral outcome in such societies. This condition establishes that the electoral competitions studied in this paper implement a social choice rule in which the society's mean log-likelihood probabilistic choice function is maximized. This theorem implies a number of interesting results on the nature of the set of electoral outcomes (Corollary 1), on the existence of electoral equilibria (Corollary 2), and on the interpretation of the main result in special cases of the model (Corollaries 3 and 4).

Appendix

Lemma 1: The candidates' payoff function is given by

$$P(\Psi_1,\Psi_2) = \int_{\Theta} \frac{f(\Psi_1;\theta)}{f(\Psi_1;\theta) + f(\Psi_2;\theta)} \cdot \hat{g}(\theta) \cdot d\theta$$

Proof: By (1) and (2),

$$P_\theta^1(\Psi_1,\Psi_2) = \frac{f(\Psi_1;\theta)}{f(\Psi_1;\theta) + f(\Psi_2;\theta)}$$

for every $\theta \in \Theta$ and $(\Psi_1,\Psi_2) \in X \times X$.

The expected plurality for candidates from an individual with the probabilistic voting density function $f(x;\theta)$ is

$$P_\theta(\Psi_1,\Psi_2) = P_\theta^1(\Psi_1,\Psi_2) - P_\theta^2(\Psi_1,\Psi_2) = 2P_\theta^1(\Psi_1,\Psi_2) - 1$$

Therefore, the expected plurality for candidate 1 from the entire population is

$$P(\Psi_1,\Psi_2) = \int_{\Theta} P_\theta(\Psi_1,\Psi_2) \cdot \hat{g}(\theta) \cdot d\theta$$

$$= \int_{\Theta} \{2 \frac{f(\Psi_1;\theta)}{f(\Psi_1;\theta) + f(\Psi_2;\theta)} - 1\}\hat{g}(\theta) \cdot d\theta$$

Since $f(x; \theta)$ is a positive, bounded, continuous function of θ for each x, $P_\theta(\Psi_1,\Psi_2)$ is continuous and bounded by -1 and +1. Therefore, since $\hat{g}(\theta)$ is a continuous density function, $P(\Psi_1,\Psi_2)$ is defined as a Riemann integral. Q.E.D.

The directional derivative of $P(x,y^*)$, with respect to changes in candidate 1's strategy, at x in the direction u is defined to be

$$D_u P(x,y^*) = \lim_{\lambda \to 0} \frac{P(x + \lambda u, y^*) - P(x,y^*)}{\lambda}$$

whenever this limit exists.

A direction, $u \in R^n$, is a <u>feasible direction</u> for candidate 1 at (x,y^*) if and only if there exists some scalar, $v_1 > 0$, such that $x + v \cdot u \in X$ for every $v \in (0,v_1)$.

The feasible directions and the directional derivative, $D_v P(x^*,y)$, for candidate 2 are analogously defined.

<u>Lemma 2</u>: (x^*,y^*) is an <u>electoral equilibrium</u> if and only if

(2.1) $D_u P(x,y^*) \le 0$ at $x = x^*$

(2.2) $D_v P(x^*,y) \ge 0$ at $y = y^*$

for all feasible directions u and v.

<u>Proof</u>: We will first show that, for any $\theta \in \Theta$,

$$H(x) = \frac{f(x;\ \theta)}{f(x;\ \theta) + f(y^*;\ \theta)}$$

is a log-concave function of x. To simplify the notation here, $f(x)$ will denote $f(x;\ \theta)$ and k will denote $f(y^*;\ \theta)$. What we must show is

$$\log H(\alpha \cdot x + (1 - \alpha) \cdot z) \geq \alpha \cdot \log H(x) + (1 - \alpha) \cdot H(z)$$

for all $x,z \in X$ and $\alpha \in [0,1]$.

By the log-concavity of $f(x)$,

$$f(\alpha \cdot x + (1 - \alpha) \cdot z) \geq f(x)^{\alpha} \cdot f(z)^{1-\alpha} \quad .$$

Therefore, since $H(x)$ is a strictly monotone increasing function of $f(x)$, we have

$$\frac{f(\alpha \cdot x + (1 - \alpha) \cdot z)}{f(\alpha \cdot x + (1 - \alpha) \cdot z) + k} \geq \frac{f(x)^{\alpha} \cdot f(z)^{1-\alpha}}{f(x)^{\alpha} \cdot f(z)^{1-\alpha} + k} \quad .$$

Since,

$$f(x)^{\alpha} \cdot f(z)^{1-\alpha} + k \leq [f(x) + k]^{\alpha} \cdot [f(z) + k]^{1-\alpha}$$

we have

$$\frac{f(\alpha \cdot x + (1 - \alpha) \cdot z)}{f(\alpha \cdot x + (1 - \alpha) \cdot z) + k} \geq \frac{f(x)^{\alpha} \cdot f(z)^{1-\alpha}}{[f(x) + k]^{\alpha} \cdot [f(z) + k]^{1-\alpha}} \quad .$$

Therefore, since $\log(w)$ is a strictly monotone increasing function of w,

$$\log \frac{f(\alpha \cdot x + (1 - \alpha) \cdot z)}{f(\alpha \cdot x + (1 - \alpha) \cdot z) + k} \geq \alpha \cdot \log \frac{f(x)}{f(x) + k} + (1 - \alpha) \cdot \log \frac{f(z)}{f(z) + k}$$

Hence, $H(x)$ is log-concave.

We will now show that, for any $\theta \in \Theta$,

$$G(x) = \int_{\Theta} \frac{f(x;\, \theta)}{f(x;\, \theta) + f(y^*;\, \theta)} \cdot \hat{g}(\theta) \cdot d\theta$$

is a log-concave function of x.

Consider sums of the form

$$\sum_{k=1}^{m} \frac{f(x;\, \theta_k)}{f(x;\, \theta_k) + f(y^*;\, \theta_k)} \cdot \hat{g}(\theta_k) \cdot \mu(I_k)$$

where (i) $I_1, \ldots, I_m$ os a partition of x (ii) $\theta_k \in I_k$, and (iii) μ is the Lebesgue measure.

Since each $[f(x;\, \theta_k)]/[f(x;\, \theta_k) + f(y^*;\, \theta_k)]$ is log-concave and each $\hat{g}(\theta_k) \cdot \mu(I_k)$ is a non-negative real number, each term in this sum is log-concave function of x (Roberts and Varberg, Theorem 13.F). Additionally the limit,

$$\lim_{m \to \infty} \sum_{k=1}^{m} \frac{f(x;\, \theta_k)}{f(x;\, \theta_k) + f(y^*;\, \theta_k)} \cdot \hat{g}(\theta_k) \cdot \mu(I_k) \quad ,$$

which is $G(x)$, exists and is positive. Therefore, since each sum in this sequence is log-concave, $G(x)$ is a log-concave function of x (Roberts and Varberg, Theorem 13.F).

Finally, since $G(x)$ is log-concave, x^* is a global maximum of $G(x)$ if and only if

$$D_u G(x) \leq 0 \quad \text{at} \quad x = x^*$$

for every feasible direction u. Therefore, since $P(x,y^*) = 2 \cdot G(x) - 1$ (where $G(x)$ is defined with $f(y^*; \theta)$, as above), and X is convex, (2.1) follows.

(2.2) follows by an analogous argument for candidate 2 (by the symmetry of $P(x,y)$). Q.E.D.

Lemma 3: x^* is a global maximum of $L(x)$ if and only if

$$D_u L(x) \leq \quad \text{at} \quad x = x^*$$

for every feasible direction u.

Proof: To establish this Lemma, we will show that $L(x)$ is concave.

First, $\log (f(x; \theta))$ is a bounded, continuous function of x since $f(x; \theta)$ has these properties. Therefore, since $\hat{g}(\theta)$ is a continuous density function, $L(x)$ is defined as a Riemann integral.

Secondly, $\log (f(x; \theta))$ is concave (since $f(x; \theta)$ is log-concave). Therefore,

$$\begin{aligned}
& L(\alpha \cdot x + (1 - \alpha) \cdot y) \\
&= \int_\Theta \log (f(\alpha \cdot x + (1 - \alpha) \cdot y; \theta)) \cdot \hat{g}(\theta) \cdot d\theta \\
&\geq \int_\Theta [\alpha \cdot \log f(x; \theta) + (1 - \alpha) \cdot \log f(y; \theta)] \cdot \hat{g}(\theta) \cdot d\theta \\
&= \alpha \cdot L(x) + (1 - \alpha) \cdot L(y)
\end{aligned}$$

The lemma now follows since $L(x)$ is concave. Q.E.D.

Proof of Theorem 1: Since $P(x,y)$ is a symmetric payoff function, Ψ is an electoral outcome (i.e., the strategy of one of the candidates in an electoral equilibrium) if, and only if, (Ψ,Ψ) is an electoral equilibrium.

We can therefore complete the proof of Theorem 1 (using Lemma 2 and Lemma 3) by showing that

$$D_u P(x,\Psi) \leq 0 \quad \text{at} \quad x = \Psi$$

if and only if

$$D_u L(x) \leq 0 \quad \text{at} \quad x = \Psi$$

for every feasible direction u.

Recall that

$$D_u P(x,\Psi) = \nabla P(x,\Psi) \cdot u \quad \text{at} \quad x = \Psi$$

for any feasible direction u (e.g., Apostol, Theorem 6-13). To obtain $\nabla P(x,\Psi)$ at $x = \Psi$, let us find the partial derivatives

$$\left.\frac{\partial P(x,\Psi)}{\partial x_h}\right]_{x=\Psi}$$

for $h = 1,\ldots,n$. In particular, we have

$$\left.\frac{\partial P(x;\theta)}{\partial x_h}\right]_{x=\Psi} = \int_{\Theta} \frac{\partial}{\partial x_h}\left(\frac{2 \cdot f(x;\theta)}{f(x;\theta) + f(\Psi;\theta)} - 1\right)\Bigg]_{x=\Psi} \hat{g}(\theta) \cdot d\theta \qquad \underline{1}/$$

$$= 2\int_{\Phi} \frac{f(\Psi;\,\theta)[\partial f(x;\,\theta)/\partial x_h]}{[f(x;\,\theta)+f(\Psi;\,\theta)]^2}\Bigg]_{x=\Psi} \hat{g}(\theta)\cdot d\theta$$

$$= \int_{\Phi} \frac{\partial f(x;\,\theta)/\partial x_h]_{x=\Psi}}{2f(\Psi;\,\theta)} \hat{g}(\theta)\cdot d\theta \quad \underline{2}/$$

$$= \frac{1}{2}\,\frac{\partial L(x)}{\partial x_h}\Bigg]_{x=\Psi}$$

Therefore

$$D_u P(x,\Psi) = \nabla P(x,\Psi)\cdot u = \frac{1}{2}\,L(x)\cdot u = \frac{1}{2}D_u L(x) \quad \text{at} \quad x = \Psi$$

for every permissible direction u. Q.E.D.

Proof of Corollary 1: Since $L(x)$ is concave, and X is is convex, the set of maxima of $L(x)$ is convex. Q.E.D.

Proof of Corollary 2: Since $f(x;\,\theta)$ is a continuous function of x, $L(x)$ is a continuous function of x by the Legesgue Dominated Convergence Theorem. Since X is compact there exists a maximum for $L(x)$. Any such maximum is an electoral outcome (by Theorem 1). Q.E.D.

Proof of Corollary 3: Follows directly from the theorem, the definition of cardinal probabilistic voting and the definition of $W(x)$. Q.E.D.

Proof of Corollary 4: When $\hat{g}(\theta)$ is the discrete density function $g(\theta_i) = \frac{1}{m}$ for every $i = 1,\ldots,m$, we get that any $x \in X$ is an electoral outcome if, and only if, x is a maximum of

$$\frac{1}{m} \cdot \sum_{i=1}^{m} \ln f(x;\, \theta_i) = \frac{1}{m} \ln \prod_{i=1}^{m} f(x;\, \theta_i) = \frac{1}{m} \ln L'(x) \quad .$$

Since $\ln(z)$ is a strictly monotone increasing function, any maximum of $\frac{1}{m} \ln L'(x)$ is also a maximum of $L'(x)$. Q.E.D.

Footnotes

1/ By Corollary 5.9 in Bartle [1966] and the regularity conditions on the $f(x; \theta)$.

2/ By Corollary 5.9 in Bartle [1966] and the regularity conditions on the $f(x; \theta)$.

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300. "Electoral Outcomes and Social Log-Likelihood Maxima," by Peter Coughlin and Shmuel Nitzan.